Summary & extra notes for substitutions

In class we saw three types of substitutions:

($f(\cdot)$ is just some expression)

1) Eqs of the form $\dfrac{dy}{dx} = f(ax+by+c)$ (no specific name)

use sub $v = ax+by+c$

2) Eqs of the form $\dfrac{dy}{dx} = f(y/x)$ ("homogeneous eq")

use sub $v = y/x$

3) Eqs of the form $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$ ("Bernoulli Eq")

use* sub $v = y^{1-n}$

* If $n=0$ or $n=1$, eq is also 1^{st} order linear, and maybe even separable, so other methods are valid. Using sub $v = y^{1-n}$ still works though!

Some further notes

★ Equations can fall into multiple categories of the above, as well as being separable and/or linear

Ex $\left\{ y' + \dfrac{2x}{x^2+1}y = 2x \right\}$ is 1-o linear and Bernoulli ($n=0$)

Ex $\left\{ x^2y' - 3xy = y^2 \right\} \longrightarrow \left\{ y' - \frac{3}{x}y = \frac{y^2}{x^2} \right\}$

- Bernoulli: $y' + P(x)y = Q(x)y^2$ $\quad P(x) = -3/x$, $Q(x) = 1/x^2$
- Homogeneous: $y' = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)$

A trick for checking if eq is homogeneous

An eq. is homogeneous if and only if replacing
$\begin{cases} x \to cx \\ y \to cy \end{cases}$ "doesn't change the eq." For illustration:

$$x^2 \frac{dy}{dx} - 3xy = y^2$$

First, note that because constants pass through derivatives, $\frac{dy}{dx} \to \frac{d(cy)}{d(cx)} = \frac{c}{c}\frac{dy}{dx} = \frac{dy}{dx}$.

Then

$$x^2 \frac{dy}{dx} - 3xy = y^2$$

$$\to (cx)^2 \frac{dy}{dx} - 3(cx)(cy) = (cy)^2$$

$$\to \cancel{c^2} x^2 \frac{dy}{dx} - 3\cancel{c^2} xy = \cancel{c^2} y^2$$

$$x^2 \frac{dy}{dx} - 3xy = y^2$$

so there was no change, meaning eq is homog.

We confirmed this also by writing it as

$$y' = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)$$

A few examples putting eqs into our 3 types above

<u>Ex</u> $\{(x+y)\,y' = x-y\}$

$$y' = \frac{x-y}{x+y}\,\frac{1/x}{1/x} = \frac{1-(y/x)}{1+(y/x)},$$

so this is a <u>homogeneous</u> eq.

It's not Bernoulli b/c we cant split the fraction $\frac{x-y}{x+y}$ into something like $P(x)y + Q(x)y^n$

<u>Ex</u> $\{2xy\,y' = x^2 + 2y^2\}$

$$y' = \frac{x^2+2y^2}{2xy} \cdot \frac{1/x^2}{1/x^2}$$

$$= \frac{1+2(y/x)^2}{2(y/x)} = f(y/x),$$

So eq <u>is</u> homogeneous. On the other hand,

$$y' = \frac{x^2+2y^2}{2xy} = \frac{x}{2y} + \frac{y}{x} = \frac{1}{x}y + \frac{1}{2x}y^{-1}$$

$$\Longrightarrow \quad y' - \frac{1}{x}y = \frac{1}{2x}y^{-1} \qquad y' + Py = Q\,y^{-1}$$

so the eq is Bernoulli too.

This means either sub $v = \frac{y}{x}$ or $v = y^{1-n} = y^2$ works

<u>Ex</u> $\{xy' = y + 2\sqrt{xy}\}$

$$y' = y/x + 2\,\frac{x^{1/2}y^{1/2}}{x}$$

$$y' = \frac{y}{x} + 2\left(\frac{y}{x}\right)^{1/2} \qquad y' = f(y/x)$$

so eq. is homogeneous. Also,

$$y' = \frac{1}{x}y + \frac{2}{x^{1/2}}y^{1/2}$$

$$\Rightarrow y' - \frac{1}{x}y = \frac{2}{x^{1/2}}y^{1/2} \qquad y' + Py = Qy^{1/2}$$

so eq is Bernoulli too. Again, either sub $v = y/x$ or $v = y^{1-1/2} = y^{1/2}$ work.

<u>Ex</u>: $\left\{y^2\frac{dy}{dx} + 2xy^3 = 6x\right\}$

$$\frac{dy}{dx} + 2xy = 6xy^{-2} \qquad y' + Py = Qy^{-2}$$

So this is Bernoulli. On the other hand, it is <u>not</u> homogeneous. We can verify this with the "trick" above.

Changing $\begin{cases} x \to cx \\ y \to cy \end{cases}$, we get

$$\frac{dy}{dx} + 2xy = 6xy^{-2}$$

$$\longrightarrow \quad \frac{dy}{dx} + 2c^2xy = 6c\cdot c^{-2}xy^{-2}$$

$$\frac{dy}{dx} + 2c^2xy = 6\frac{1}{c}xy^{-2}.$$

Certainly this isn't the same as the "original," so eq is <u>not</u> homogeneous.