

Summary & extra notes for substitutions

In class we saw three types of substitutions:

1) Eqs of the form $\frac{dy}{dx} = f(ax+by+c)$
use sub $v = ax+by+c$ (no specific name)
($f(\cdot)$ is just some expression)

2) Eqs of the form $\frac{dy}{dx} = f(y/x)$ ("homogeneous eq")
use sub $v = y/x$

3) Eqs of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$
use* sub $v = y^{1-n}$ ("Bernoulli eq")
also

* If $n=0$ or $n=1$, eq is 1^{st} order linear, and maybe even separable, so other methods are valid. Using sub $v = y^{1-n}$ still works though!

Some further notes

★ Equations can fall into multiple categories of the above, as well as being separable and/or linear

Ex $\{ y' + \frac{2x}{x^2+1}y = 2x \}$ is 1st order linear and Bernoulli ($n=0$)

Ex $\{ x^2y' - 3xy = y^2 \} \rightarrow \{ y' - \frac{3}{x}y = \frac{y^2}{x^2} \}$

- Bernoulli: $y' + P(x)y = Q(x)y^2$ $P(x) = -3/x$
 $Q(x) = 1/x^2$
- Homogeneous: $y' = (\frac{y}{x})^2 + 3(\frac{y}{x})$

A trick for checking if eq is homogeneous

An eq. is homogeneous if and only if replacing
 $\begin{cases} x \rightarrow cx \\ y \rightarrow cy \end{cases}$ "doesn't change the eq." For illustration:

$$x^2 \frac{dy}{dx} - 3xy = y^2$$

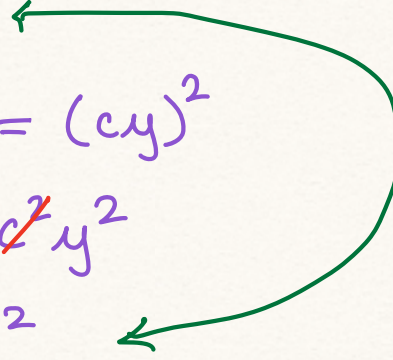
First, note that because constants pass through derivatives, $\frac{dy}{dx} \rightarrow \frac{d(cy)}{d(cx)} = \frac{c}{c} \frac{dy}{dx} = \frac{dy}{dx}$.

Then

$$x^2 \frac{dy}{dx} - 3xy = y^2$$


$$\rightarrow (cx)^2 \frac{dy}{dx} - 3(cx)(cy) = (cy)^2$$

$$\rightarrow \cancel{c^2} x^2 \frac{dy}{dx} - 3\cancel{c^2} xy = \cancel{c^2} y^2$$

$$x^2 \frac{dy}{dx} - 3xy = y^2$$


so there was no change, meaning eq is homog.

We confirmed this also by writing it as

$$y' = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)$$


A few examples putting eqs into our 3 types above

$$\underline{\text{Ex}} \{ (x+y) y' = x-y \}$$

$$y' = \frac{x-y}{x+y} \cdot \frac{1/x}{1/x} = \frac{1-(y/x)}{1+(y/x)},$$

so this is a homogeneous eq.

It's not Bernoulli b/c we can't split the fraction $\frac{x-y}{x+y}$ into something like $P(x)y + Q(x)y^n$

$$\underline{\text{Ex}} \{ 2xy y' = x^2 + 2y^2 \}$$

$$y' = \frac{x^2 + 2y^2}{2xy} \cdot \frac{1/x^2}{1/x^2}$$

$$= \frac{1 + 2(y/x)^2}{2(y/x)} = f(y/x),$$

So eq is homogeneous. On the other hand,

$$y' = \frac{x^2 + 2y^2}{2xy} = \frac{x}{2y} + \frac{y}{x} = \frac{1}{x}y + \frac{1}{2x}y^{-1}$$

$$\Rightarrow y' - \frac{1}{x}y = \frac{1}{2x}y^{-1} \quad y' + Py = Qy^{-1}$$

so the eq is Bernoulli too.

This means either sub $v = \frac{y}{x}$ or $v = y^{1-n} = y^2$ works

Ex $\{ xy' = y + 2\sqrt{xy} \}$

$$y' = y/x + 2 \frac{x^{1/2} y^{1/2}}{x}$$

$$y' = \frac{y}{x} + 2 \left(\frac{y}{x}\right)^{1/2} \quad y' = f(y/x)$$

so eq. is homogeneous. Also,

$$y' = \frac{1}{x} y + \frac{2}{x^{1/2}} y^{1/2}$$

$$\Rightarrow y' - \frac{1}{x} y = \frac{2}{x^{1/2}} y^{1/2} \quad y' + P y = Q y^{1/2}$$

so eq is Bernoulli too. Again, either sub $v = y/x$
or $v = y^{1-1/2} = y^{1/2}$ work.

Ex: $\{ y^2 \frac{dy}{dx} + 2xy^3 = 6x \}$

$$\frac{dy}{dx} + 2xy = 6xy^{-2} \quad y' + P y = Q y^{-2}$$

So this is Bernoulli. On the other hand, it is not homogeneous. We can verify this with the "trick" above.

Changing $\begin{cases} x \rightarrow cx \\ y \rightarrow cy \end{cases}$, we get

$$\frac{dy}{dx} + 2xy = 6xy^{-2}$$

$$\rightarrow \frac{dy}{dx} + 2c^2 xy = 6c \cdot c^{-2} xy^{-2}$$

$$\frac{dy}{dx} + 2c^2 xy = 6 \frac{1}{c} xy^{-2}.$$

Certainly this isn't the same as the "original,"

so eq is not homogeneous.